Hilroy

80 Pages

27.6 cm x 21.2 cm

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EXERCISE BOOK
CAHIER D'EXERCICES

NAME/NOM [illegible]

SUBJECT/SUJET MATH 253

ASSEMBLED IN CANADA WITH IMPORTED MATERIALS
ASSEMBLÉ AU CANADA AVEC DES MATIÈRES IMPORTÉES

12107

Overview

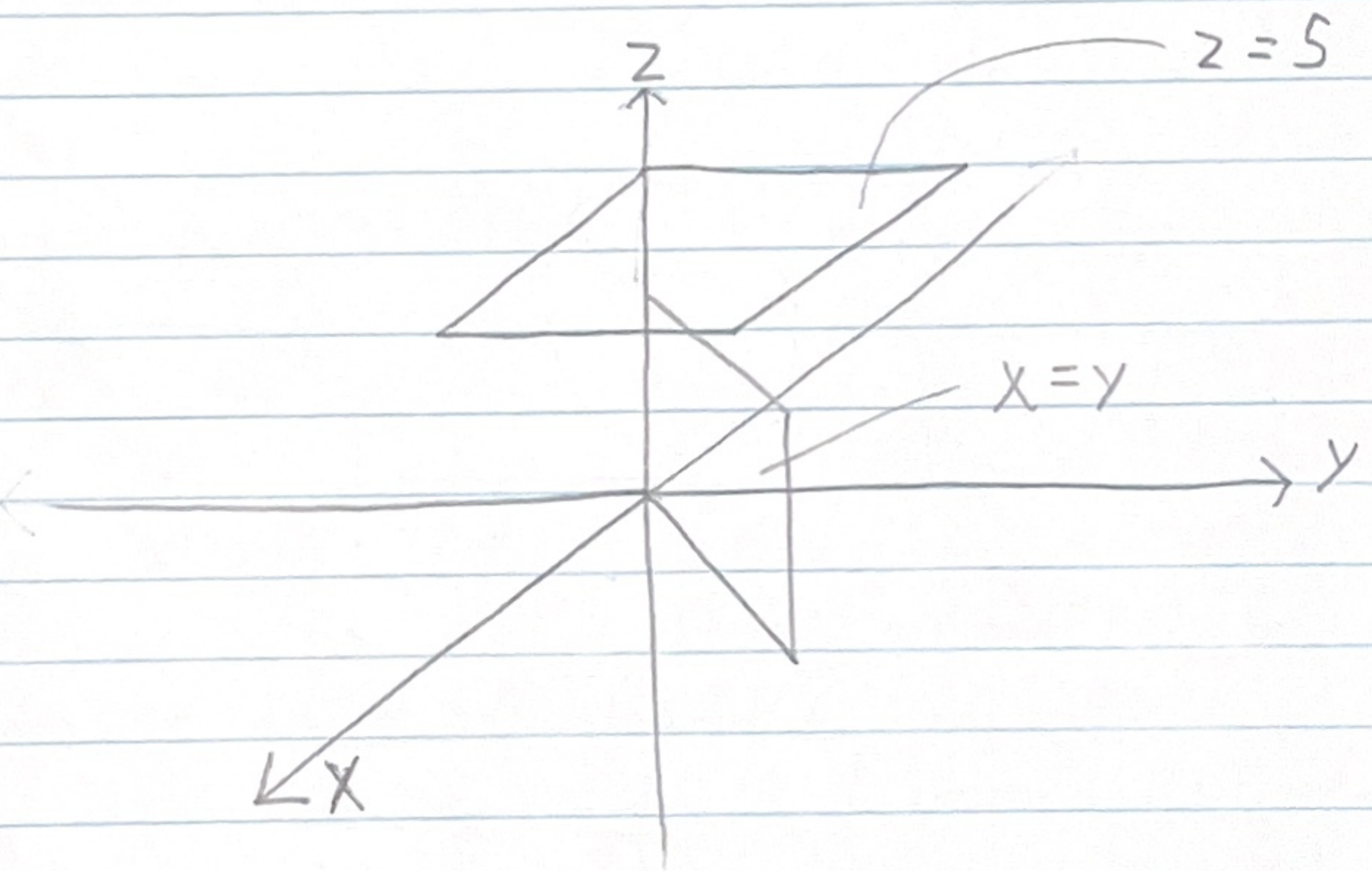
9/5/18

Office hours

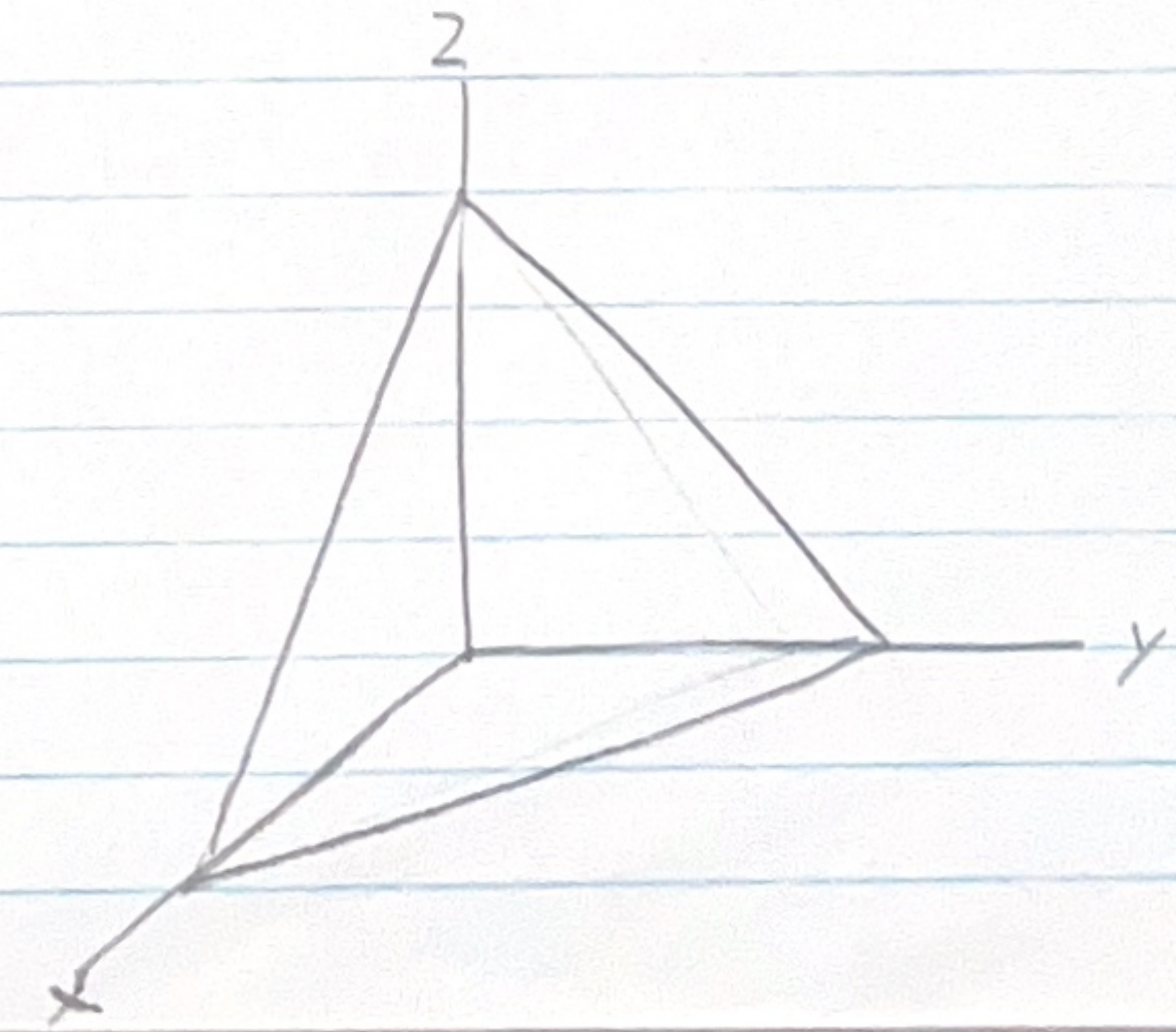
Monday 1:00-2:00
Wednesday 1:00-2:00
MATH 226

LEC 1

Standard coordinates in R^3

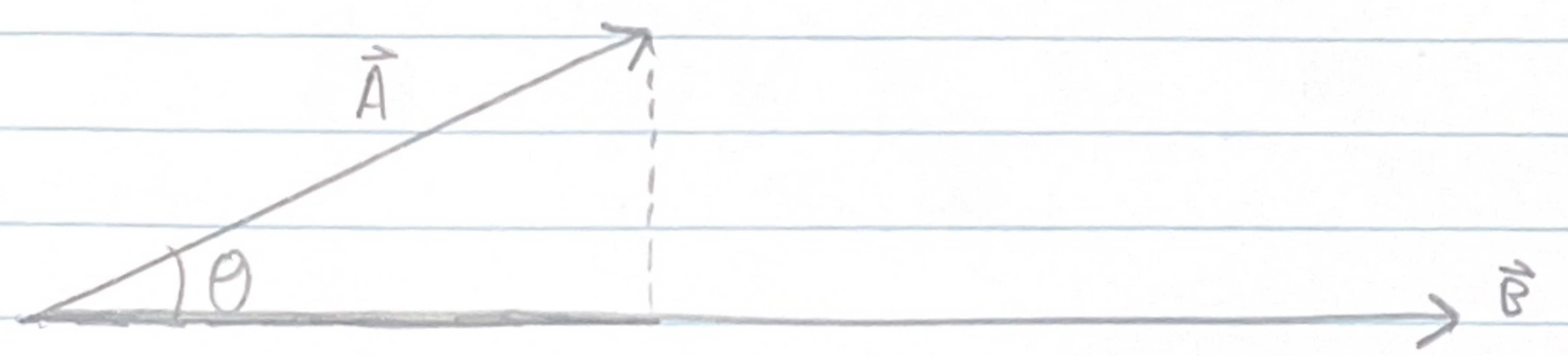


ex: $x+y+z=10$



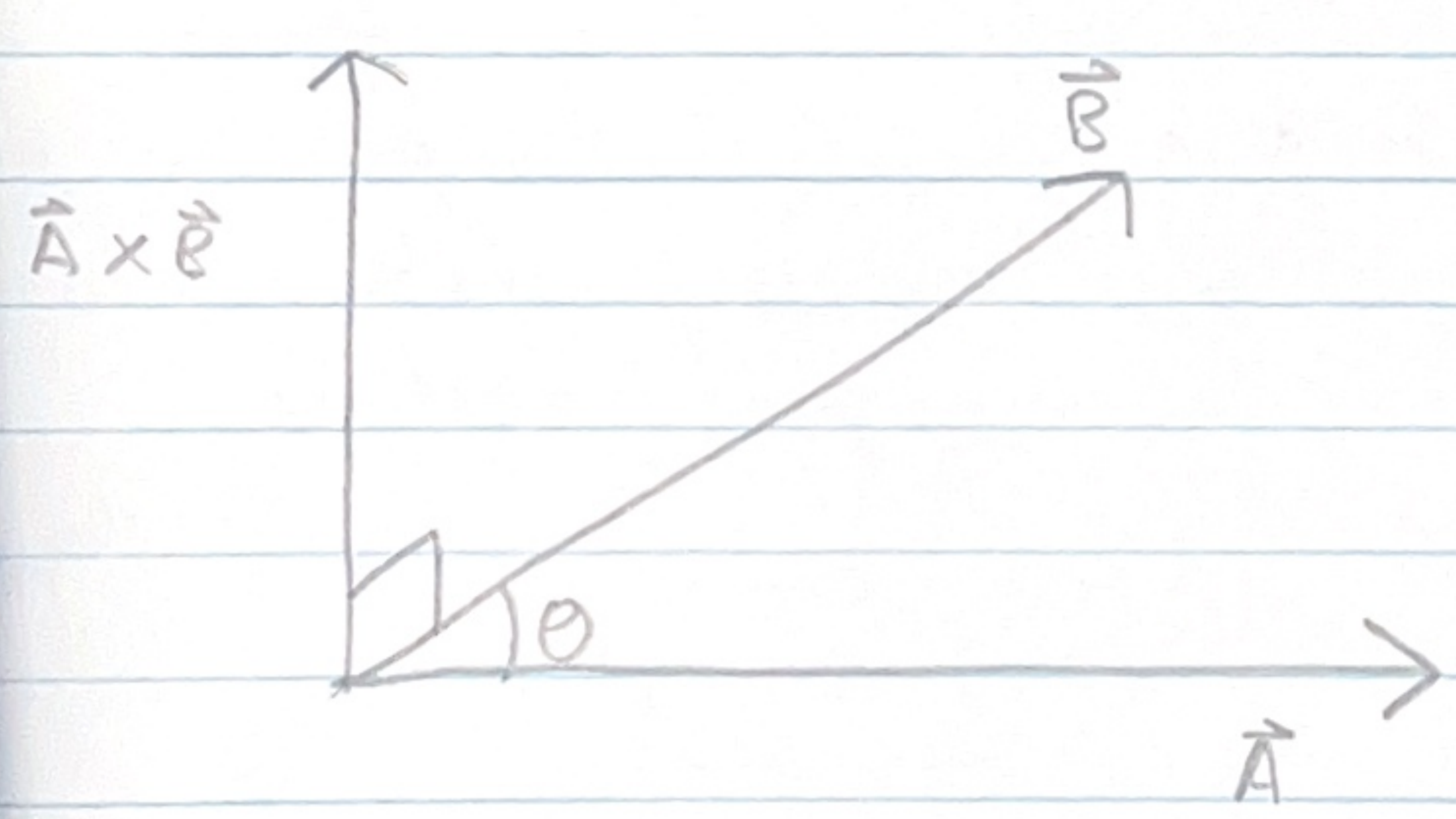
Vectors

Projections



$$\text{proj}_{\vec{B}} \vec{A} = \left(\frac{\vec{A} \cdot \vec{B}}{\vec{B} \cdot \vec{B}} \right) \vec{B}$$

Cross Product



$$|\vec{A} \times \vec{B}| = AB \sin\theta$$

$$\vec{A} \times \vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = (b_1 c_2 - c_1 b_2)\hat{i} - (a_1 c_2 - c_1 a_2)\hat{j} + (a_1 b_2 - b_1 a_2)\hat{k}$$

Eqn of a line

direction vector

$$\langle x, y, z\rangle = \langle x_0, y_0, z_0\rangle + t\langle a, b, c\rangle \quad t \in \mathbb{R}$$

$$= \langle x_0 + ta, y_0 + tb, z_0 + tc\rangle$$

$$x(t) = x_0 + ta$$
$$y(t) = y_0 + tb$$
$$z(t) = z_0 + tc$$

Example

$\langle 1, 4, -3\rangle$ and $(2, 0, 3)$

$$\langle x, y, z\rangle = \langle 2, 0, 3\rangle + t\langle 1, 4, -3\rangle \quad t \in \mathbb{R}$$

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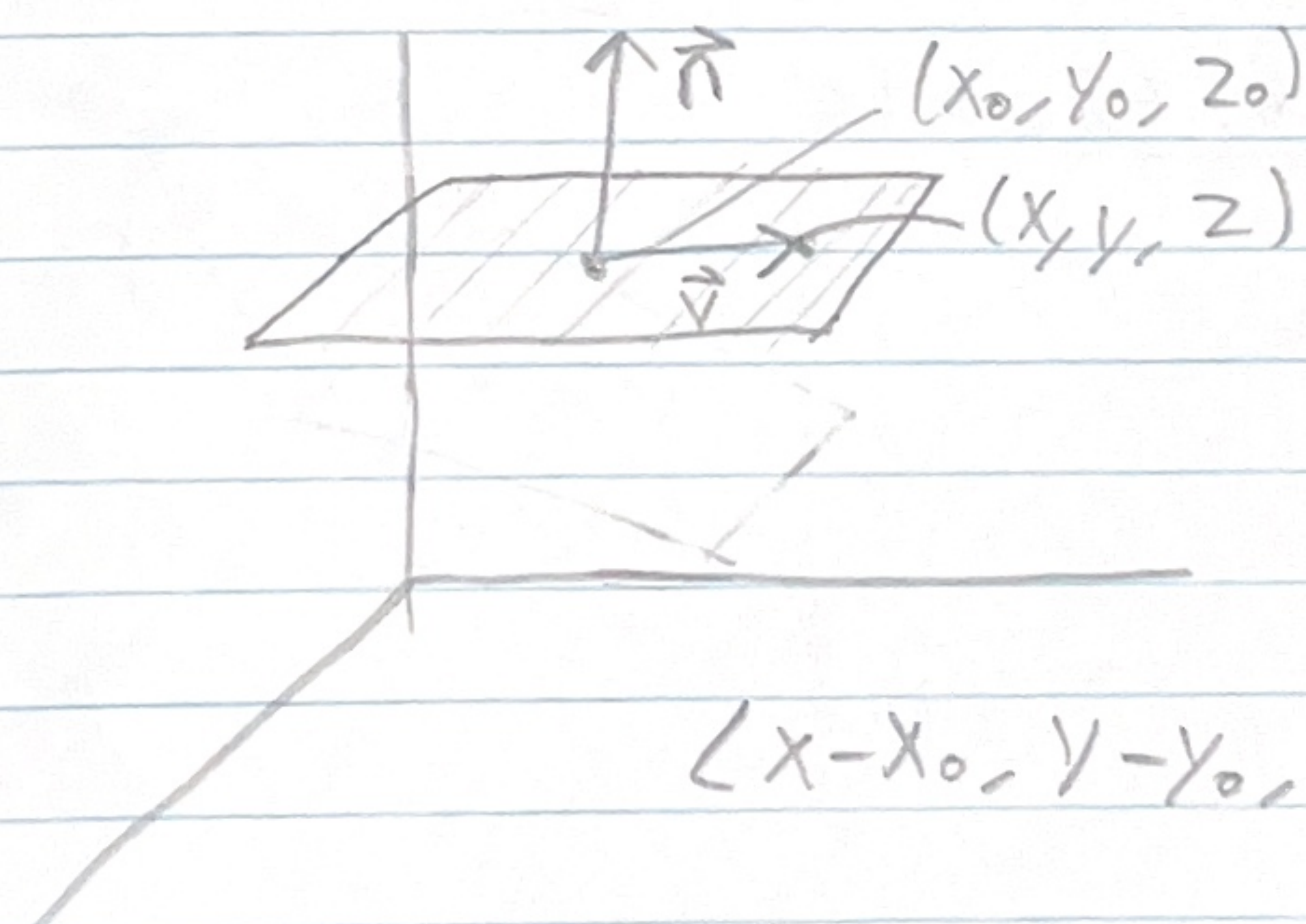
Lines & Planes



$\vec{N} = \langle a, b, c\rangle$

$\vec{v} = \langle x - x_0, y - y_0, z - z_0\rangle$

$$\langle x - x_0, y - y_0, z - z_0\rangle \cdot \langle a, b, c\rangle = 0$$

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

eqn of plane = $\boxed{ax + by + cz = d}$

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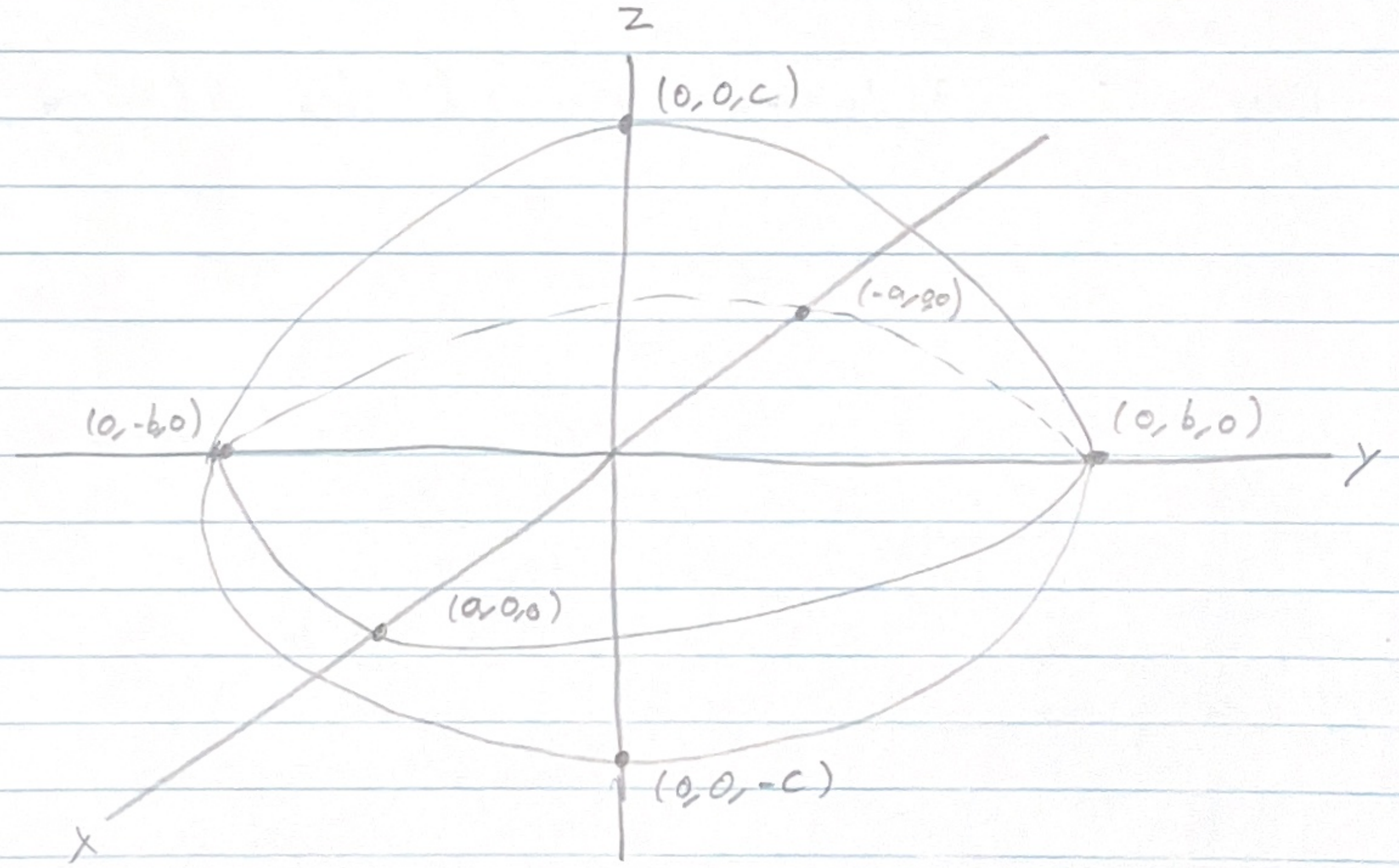
Quadric Surfaces are surfaces defined by a quadratic equation in x, y, z ie: a linear combination of $x^2, y^2, z^2, xy, yz, xz, x, y, z, 1$

- Spheres are quadric surfaces

$$(x-x_0)^2+(y-y_0)^2+(z-z_0)^2=R^2$$

$\boxed{\text{elipsoid}}$

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$



- if the equation doesn't involve one of the variables it is a Cylinder ← mathematical term

ie: $x^2+y^2=1$ ie: $z=y^2$

Functions of 2/3 Variables

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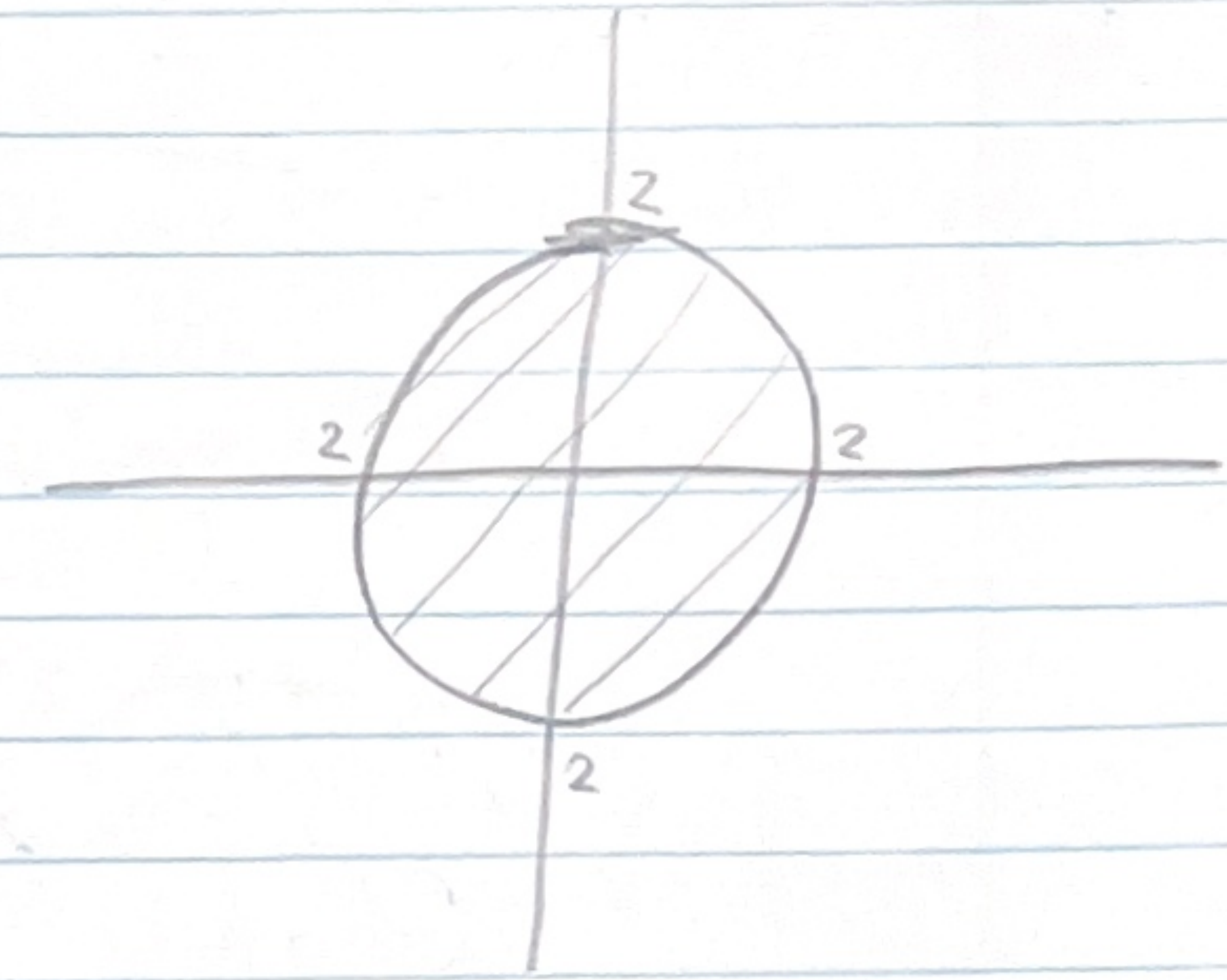
$f(x,y) = \sqrt{4-x^2-y^2}$

$x^2+y^2 \leq 4$

Domain

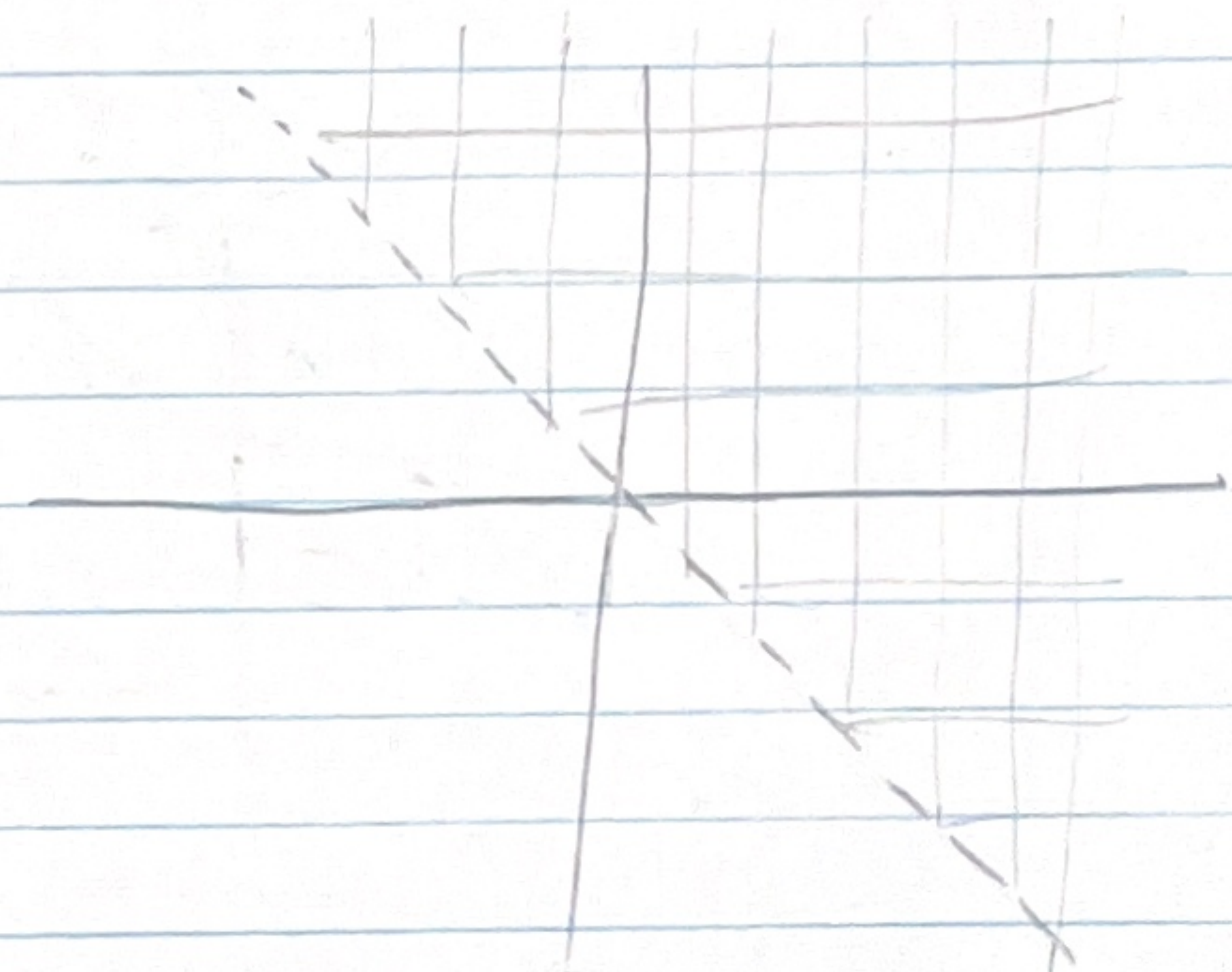


$f(x,y) = \log(x+y)$

$x+y>0$

$x>-y$

Domain

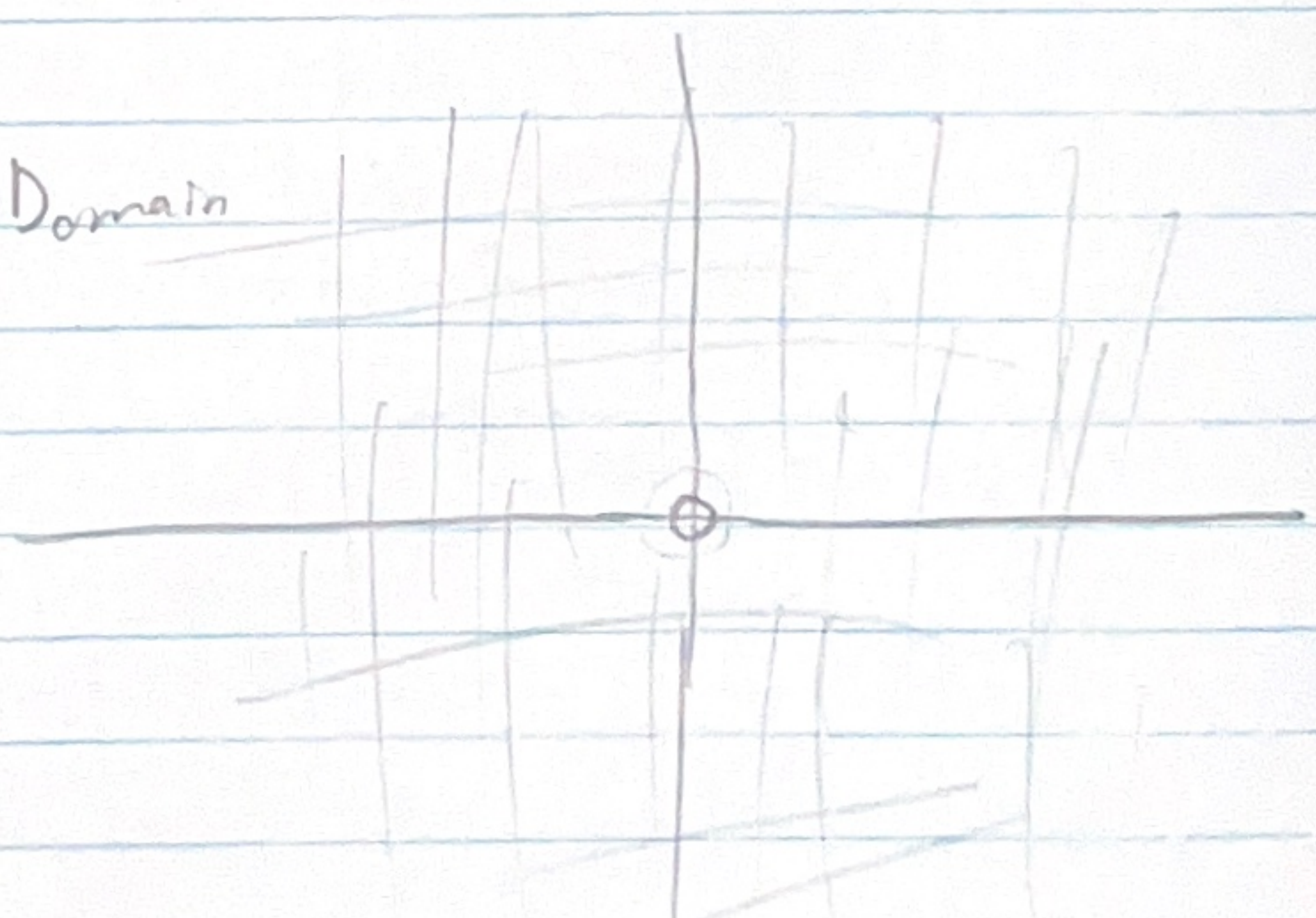


$f(x,y) = \frac{1}{x^2+y^2}$

Domain

$x^2+y^2 \neq 0$

$(0,0)$ not defined



Functions of 3 Variables

Example $f(x,y,z) = x^2+y^2+z^2$

- then the contour surfaces are concentric spheres for various k values

$k = x^2+y^2+z^2$

Contour plot

Continuity

- A one variable fcn can be discontinuous if the graph "has a tear"

- A two variable fcn $f(x,y)$ is continuous at (x_0,y_0) if $\lim_{(x,y)\to(x_0,y_0)} f(x,y) = f(x_0,y_0)$

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Continuity

$f(x,y)$ is continuous at (x_0,y_0)

if

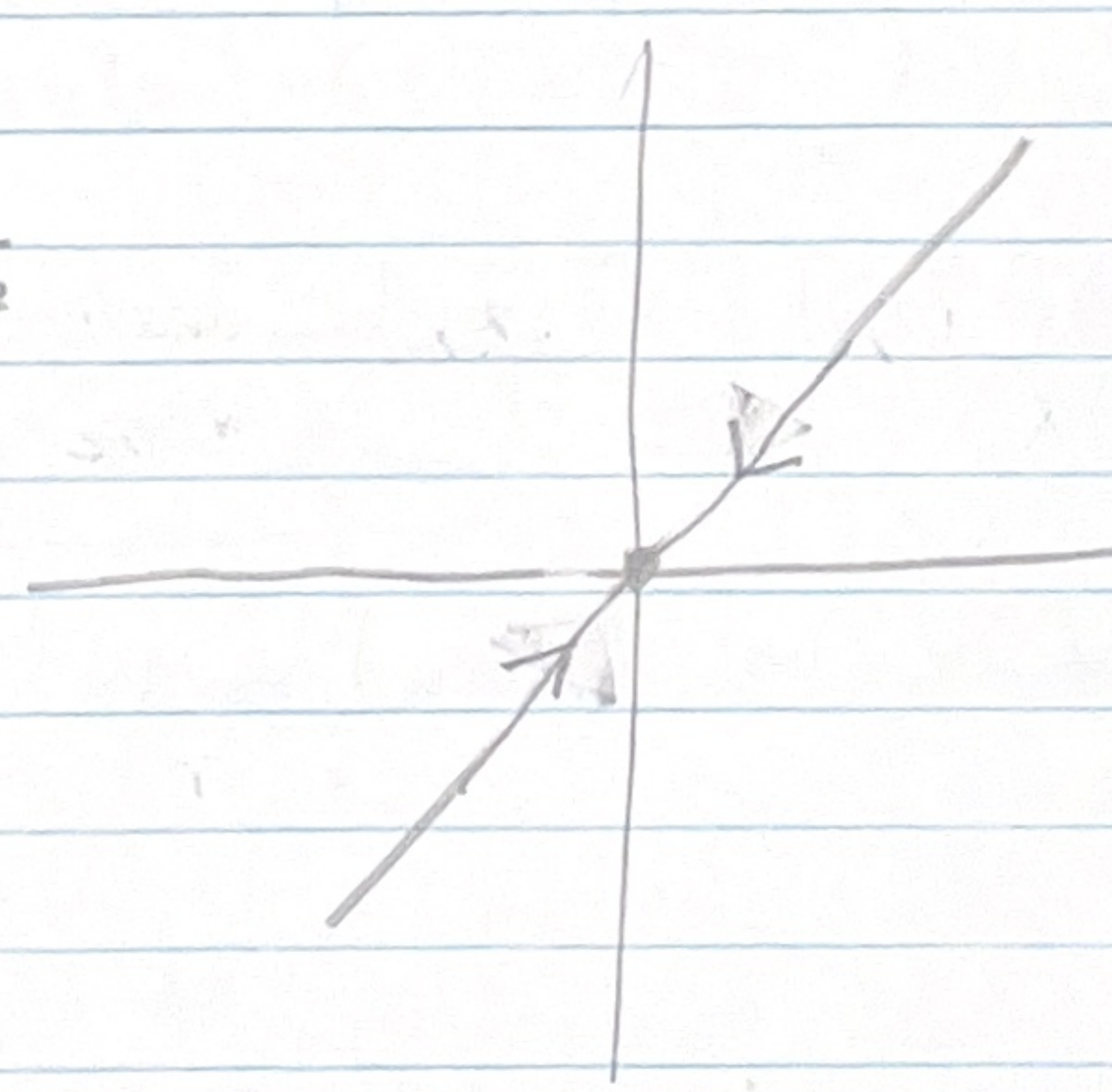
$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = f(x_0,y_0)$$

Example

$$f(x,y) = \begin{cases} \dfrac{-xy}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

is $f(x,y)$ continuous at $(0,0)$

$$\lim_{(x,y)\to(x_0,y_0)} \frac{-xy}{x^2+y^2}$$



evaluate along the
line $y=mx$

Example $f(x,y) = e^{xy^2} + \sin(x+y^3)$

$$\frac{\partial f}{\partial x} = y^2 e^{xy^2} + \cos(x+y^3)$$

$$\frac{\partial f}{\partial y} = 2xye^{xy^2} + 3y^2\cos(x+y^3)$$

Note!

$$\frac{\partial^2 f}{\partial x \partial y} = f_{xy}$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$$

$$\frac{\partial^2 f}{\partial y^2} = f_{yy}$$

Example $f_{yy} = 2xe^{xy^2} + 2xy(2xy)e^{xy^2} + 6y\cos(x+y^3) + 3y^2(3y^2)(-\sin(x+y^3))$

Interpretation in terms of slope

Partial derivatives are slopes of lines tangent to TRACE Curves

$\frac{\partial f}{\partial y}(x_0, y_0) \Rightarrow$ Slope of line tangent to $x = x_0$ Trace Curve

Partial Derivatives

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3 variables

$f(x,y,z)$ //order doesn't matter

Implicit differentiation

if we consider the relation $1 = x^2 + y^2 + z^2$

Try to make z a fcn of x,y

$$z(x,y) = \sqrt{1-x^2-y^2}$$

$$\frac{dz}{dx} = \frac{1}{2}(1-x^2-y^2)^{-1/2} \cdot (-2x)$$

$$= \frac{-x}{\sqrt{1-x^2-y^2}} = \frac{-x}{z}$$

we can get $\frac{dz}{dx}$ without solving for z first

$$1 = x^2 + y^2 + z^2$$

z is dependent
y, x are independent

$$\frac{d}{dx}(1) = \frac{d}{dx}(x^2+y^2+z^2)$$

$$0 = 2x + 0 + 2z\frac{dz}{dx}$$

$$-2x = 2z\frac{dz}{dx}$$

$$\boxed{\frac{dz}{dx} = \frac{-x}{z}}$$

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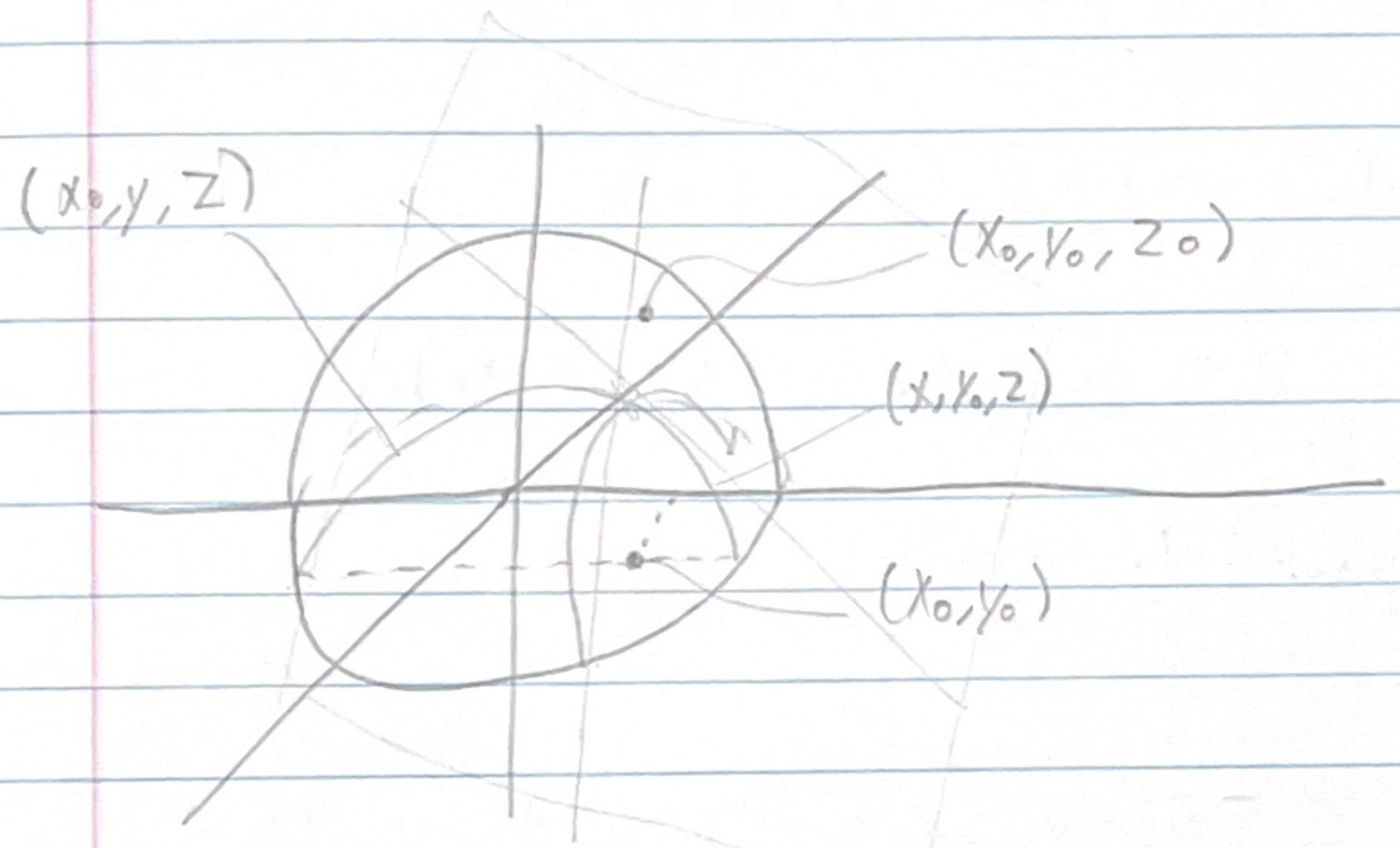
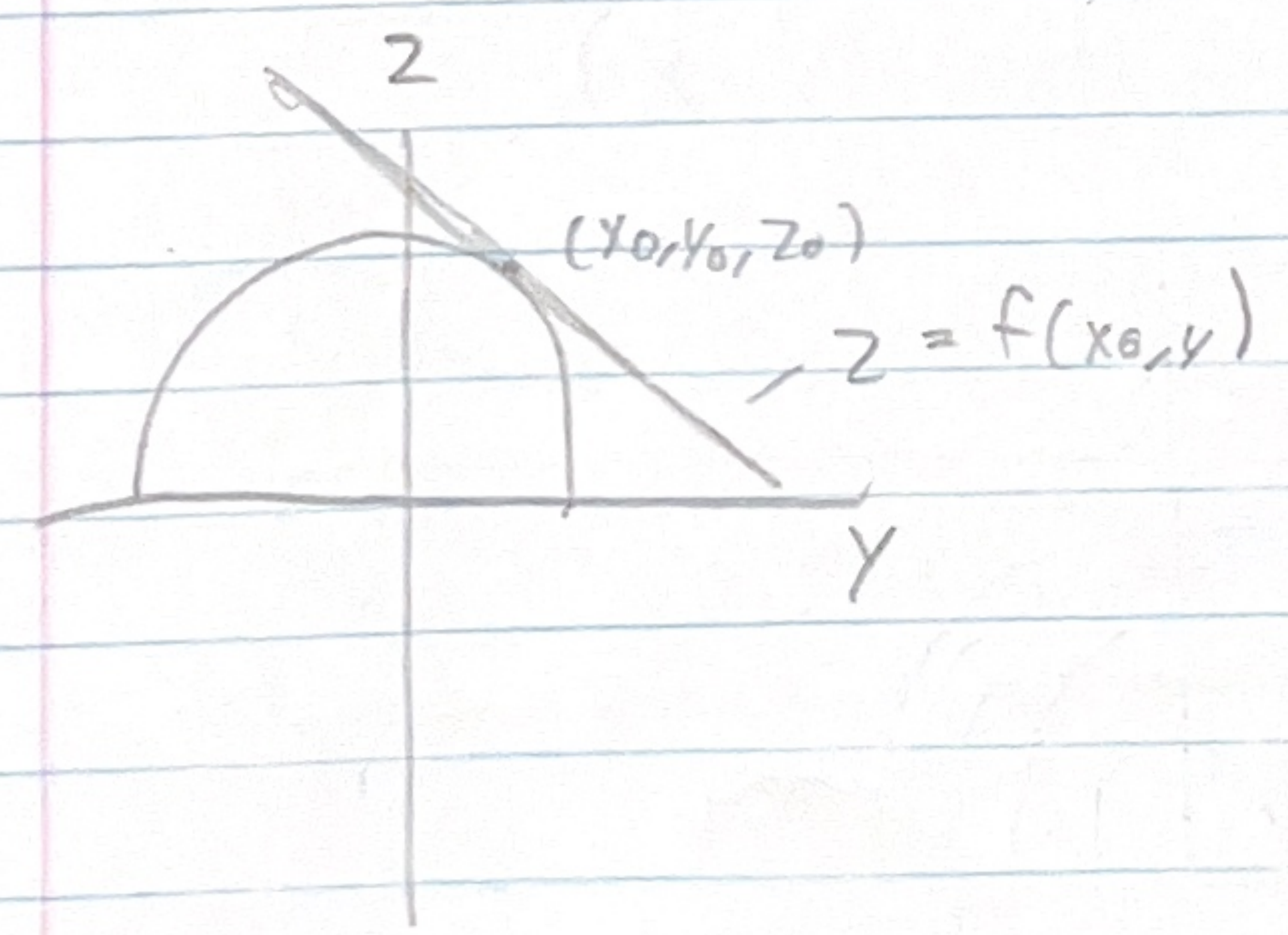
for $f(x,y)$ the best linear approximation to $f(x,y)$ near (x_0, y_0) is given by the plane tangent to $z=f(x,y)$ at $(x_0, y_0, f(x_0,y_0))$

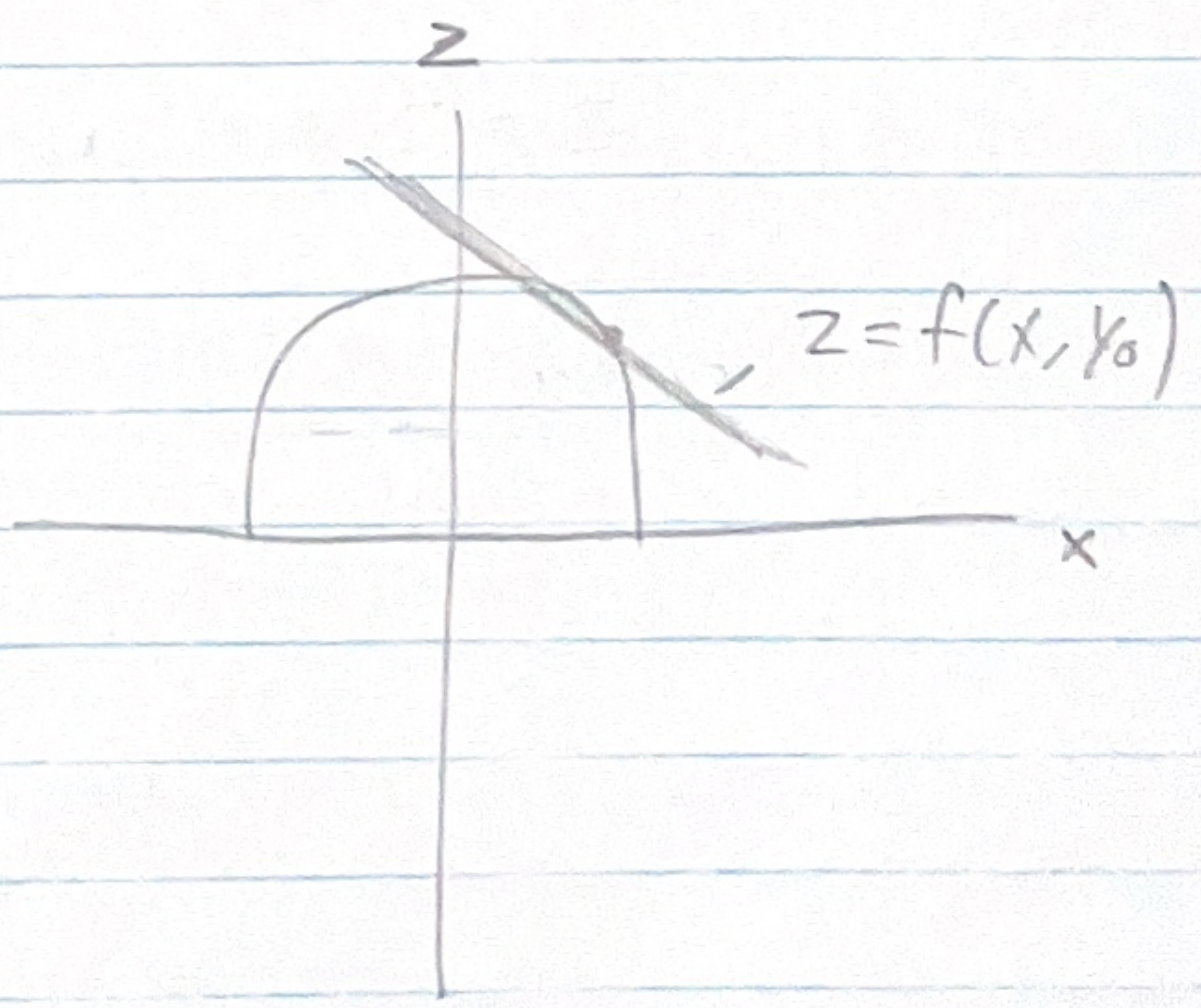
$z_0 = f(x_0, y_0)$



Plane contains two lines, which are tangents to the true trace curves for x, y

Find eqn of Plane





$z = f(x_0,y_0) + f_y(x_0,y_0)(y-y_0)$

$\uparrow$

z_0

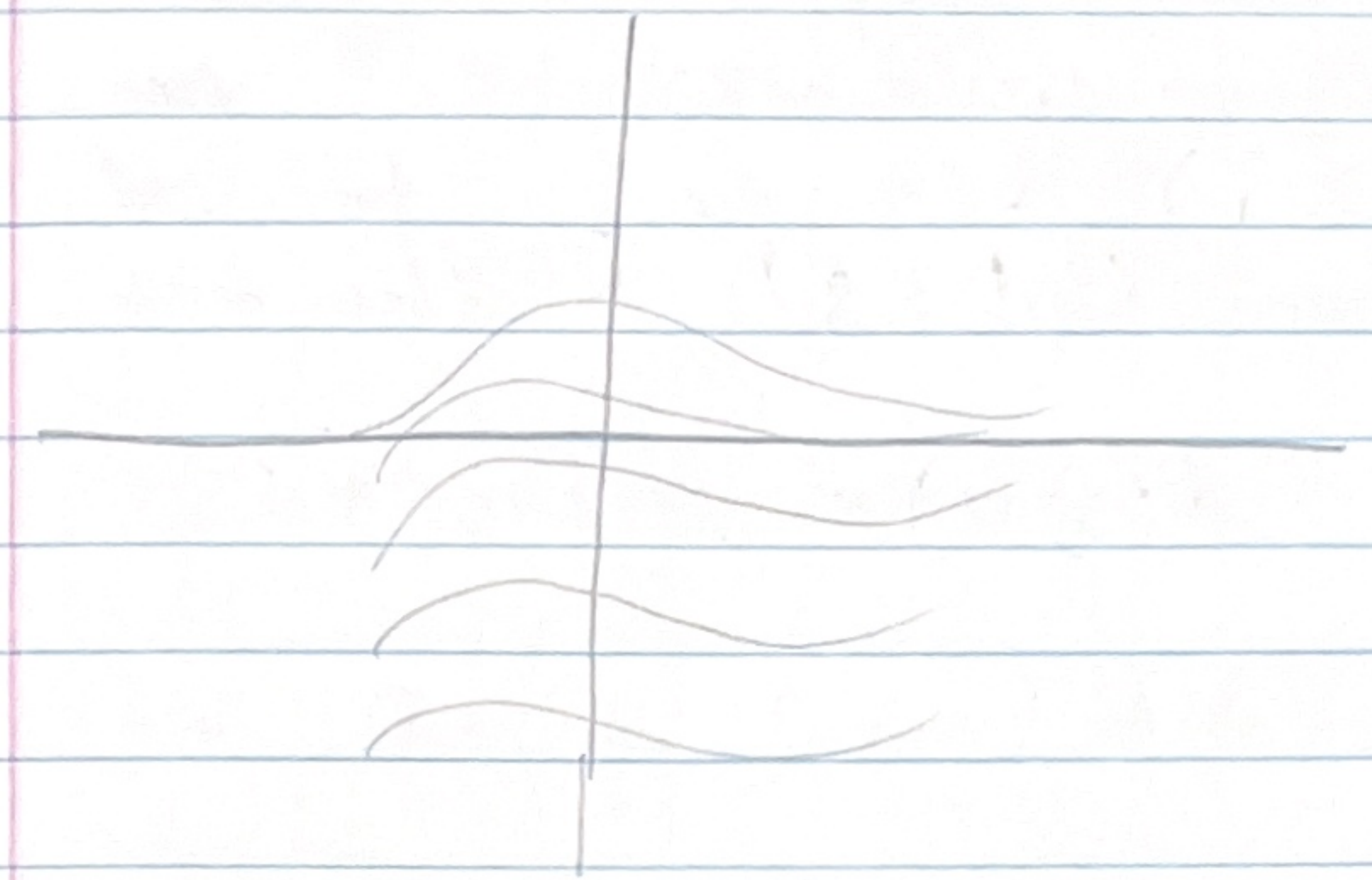
$z = f(x_0,y_0) + f_x(x_0,y_0)(x-x_0)$

$\uparrow$

z_0

Example

Find the eqn of the tangent plane to the graph $z = e^{-x^2-y^2}$ at $(x_0, y_0) = (1,1)$



$$z_0 = e^{-1^2-1^2} = e^{-2}$$

$$\frac{dz}{dx} = -2x e^{-x^2-y^2}$$

$$\frac{dz}{dy} = -2y e^{-x^2-y^2}$$

$$\left.\frac{dz}{dx}\right|_{x_0,y_0} = -2e^{-2}$$

$$\left.\frac{dz}{dy}\right|_{x_0,y_0} = -2e^{-2}$$

$$\boxed{z = e^{-2} - 2e^{-2}(x-1) - 2e^{-2}(y-1)}$$

$$e^2 z + 2x + 2y = 5$$

So $\vec{n} = \langle 2, 2, e^2 \rangle$

Linear Approximation

$$f(x,y) \approx f(x_0,y_0) + f_x(x_0,y_0)\underbrace{(x-x_0)}_{\Delta x} + f_y(x_0,y_0)\underbrace{(y-y_0)}_{\Delta y}$$

works as long as $\Delta x = x - x_0$
$\Delta y = y - y_0$
is small

$z = f(x,y) \qquad z_0 = f(x_0,y_0)$

$$z \approx z_0 + \frac{dz}{dx}(x_0,y_0)(x-x_0) + \frac{dz}{dy}(x_0,y_0)(y-y_0)$$

$\Delta z = z - z_0$

$$\boxed{\Delta z \approx \frac{dz}{dx}\Delta x + \frac{dz}{dy}\Delta y}$$

where $\left.\frac{dz}{dx}\right|_{x_0,y_0} \quad \left.\frac{dz}{dy}\right|_{x_0,y_0}$

limit as $\Delta x, \Delta y \to 0$

$$\boxed{dz = \frac{dz}{dx}dx + \frac{dz}{dy}dy}$$

differential version of linear approx

Linear approx for 3 variables

$w = f(x,y,z) \qquad w_0 = f(x_0,y_0,z_0) \qquad \Delta w = w - w_0$

$$\Delta w \approx \frac{dw}{dx}\Delta x + \frac{dw}{dy}\Delta y + \frac{dw}{dz}\Delta z$$

Multivariable Chain Rule

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if x, y are fcn's of t

$z = f(x,y) \qquad x = g(t) \quad y = h(t)$

$z = f(x(t), y(t))$

$$\boxed{\frac{dz}{dt} = \frac{dz}{dx}\frac{dx}{dt} + \frac{dz}{dy}\frac{dy}{dt}}$$

Polar Coordinates

$x = r\cos\theta$
$y = r\sin\theta$

$\frac{dx}{dr} = \cos\theta$

$\frac{dy}{dr} = \sin\theta$

$\frac{dx}{d\theta} = -r\sin\theta$

$\frac{dy}{d\theta} = r\cos\theta$

by below

$= \left(\frac{dz}{dx}\right)^2\left(\cos^2\theta + \frac{r^2\sin^2\theta}{r^2}\right) +$

$+\frac{dz}{dx}\frac{dz}{dy}\left(2\cos\theta\sin\theta - \frac{r^2 2\sin\theta\cos\theta}{r^2}\right)$

$\left(\frac{dz}{dy}\right)^2\left(\sin^2\theta + \frac{1}{r^2}r^2\cos^2\theta\right)$

Example

Compute $\left(\frac{dz}{dr}\right)^2 + \frac{1}{r^2}\left(\frac{dz}{d\theta}\right)^2$ in terms of $\frac{dz}{dx}$ and $\frac{dz}{dy}$

$\frac{dz}{dr} = \frac{dz}{dx}\frac{dx}{dr} + \frac{dz}{dy}\frac{dy}{dr}$

$= \frac{dz}{dx}\cos\theta + \frac{dz}{dy}\sin\theta$

$\frac{dz}{d\theta} = \frac{dz}{dx}\frac{dx}{d\theta} + \frac{dz}{dy}\frac{dx}{d\theta}$

$= \frac{dz}{dx}(-r\sin\theta) + \frac{dz}{dy}(r\cos\theta)$

Gradient

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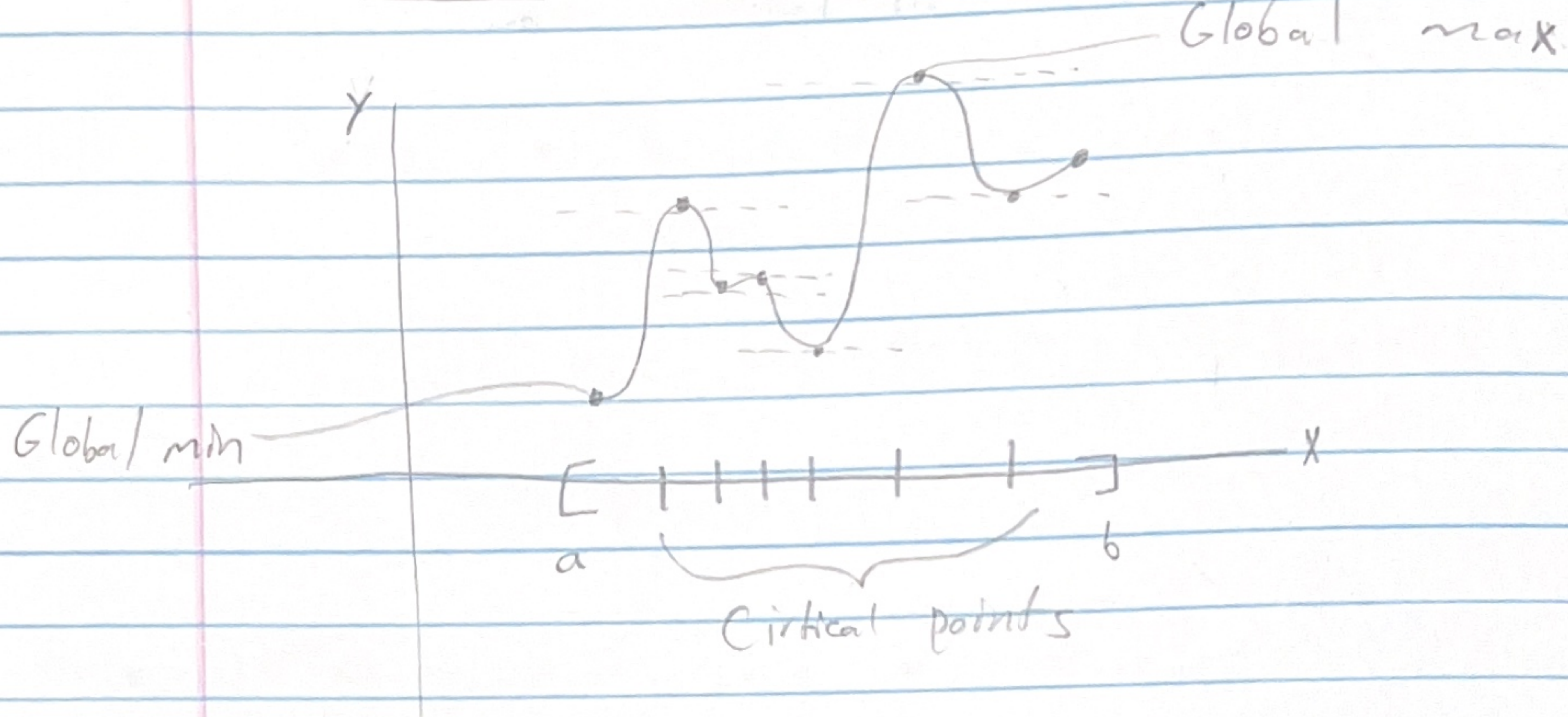
Max/Min



For R^2: (to find max/min)

- Found critical points
- looked at boundary points

@ Crit point:

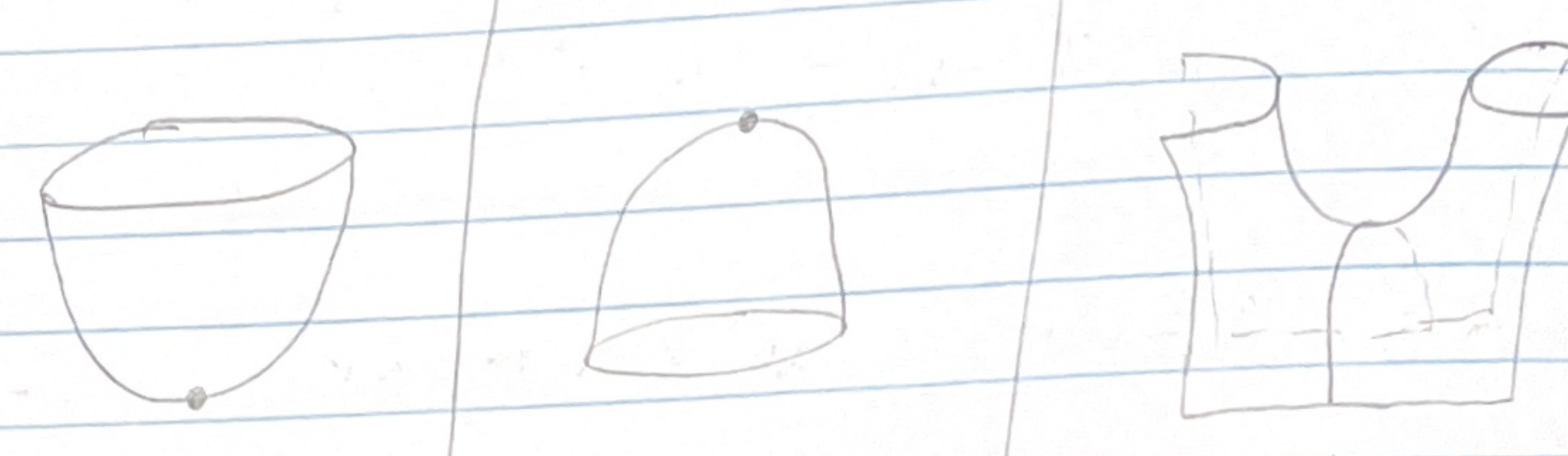
- $f''(x) > 0$ local min
- $f''(x) < 0$ local max
- $f''(x) = 0$ No Info

$D=0$, (x_0,y_0) is not an ordinary critical point

Critical Points

Ordinary critical points

local min	local max	either (saddle)



$z=x^2+y^2$	$z=-x^2-y^2$	$z=-x^2+y^2$

suppose $(\nabla f)(x_0,y_0)=0$

Then, in

$$D(x_0,y_0) = \begin{vmatrix} f_{xx}(x_0,y_0) & f_{xy}(x_0,y_0) \\ f_{yx}(x_0,y_0) & f_{yy}(x_0,y_0) \end{vmatrix}$$

$$= f_{xx}f_{yy} - f_{xy}^2$$

if

$D<0$ (x_0,y_0) is a saddle point

$D>0$ (x_0,y_0) is a local min/max

- if $D>0$ and $f_{xx}>0$, local min
- if $D>0$ and $f_{xx}<0$, local max

or f_{yy}

Max/Min of $f(x,y)$

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(x_0, y_0) is a critical point where

- $(\nabla f)(x_0, y_0) = 0$

- $D(x_0,y_0) = \begin{vmatrix} f_{xx}(x_0,y_0) & f_{xy}(x_0,y_0) \\ f_{xy}(x_0,y_0) & f_{yy}(x_0,y_0) \end{vmatrix} = f_{xx}f_{yy} - f_{xy}^2$

- $D > 0$ and $f_{xx}, f_{yy} < 0 \rightarrow$ local max
- $D > 0$ and $f_{xx}, f_{yy} > 0 \rightarrow$ local min

- $D < 0 \rightarrow$ Saddle point

- $D = 0 \rightarrow$ We don't know, not an ordinary critical point

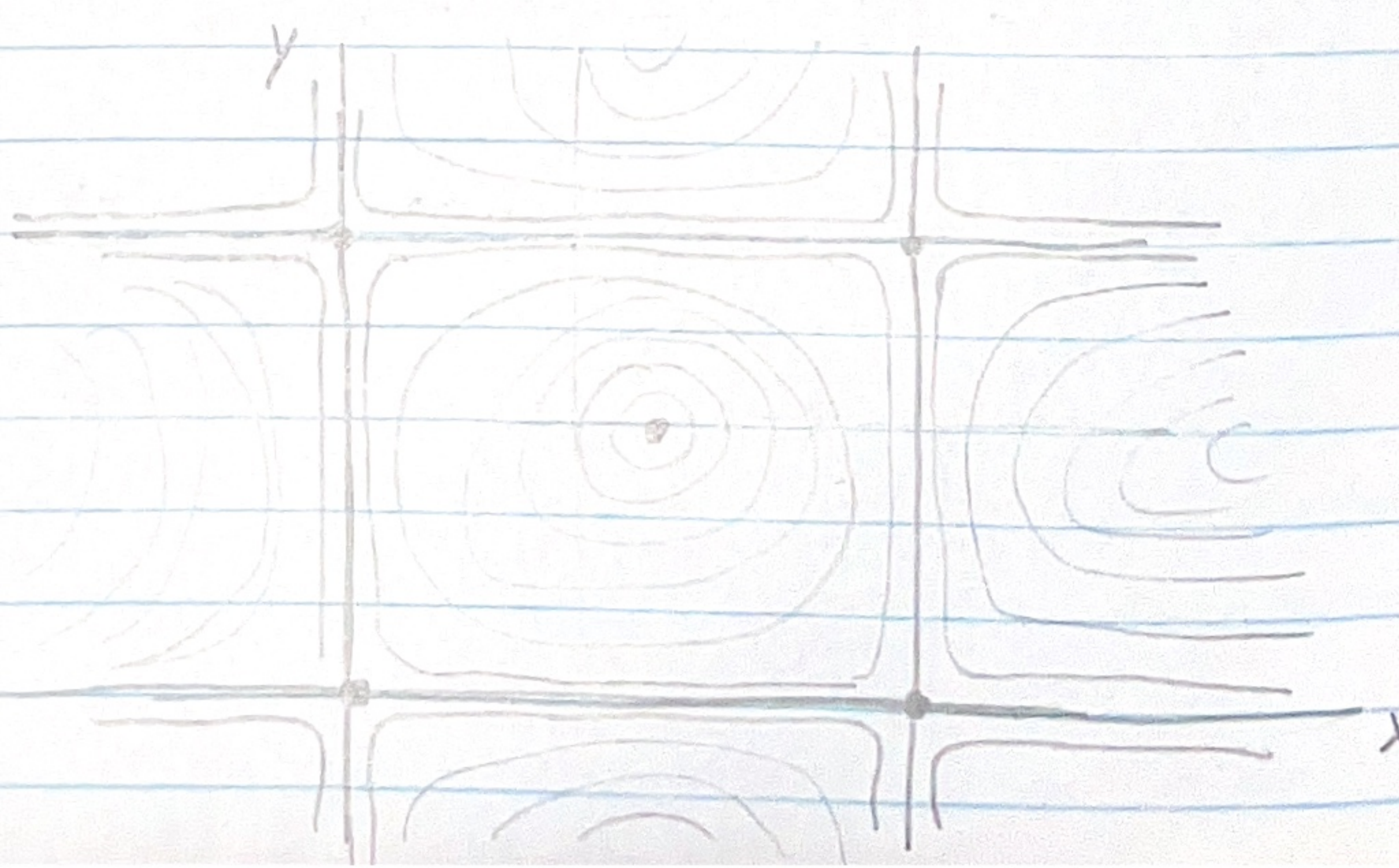
$\boxed{\text{Example}}$ $f(x,y) = (2x - x^2)(2y - y^2)$

-From last time

Crit points $\rightarrow$ $(1,1), (0,0), (0,2), (2,0), (2,2)$

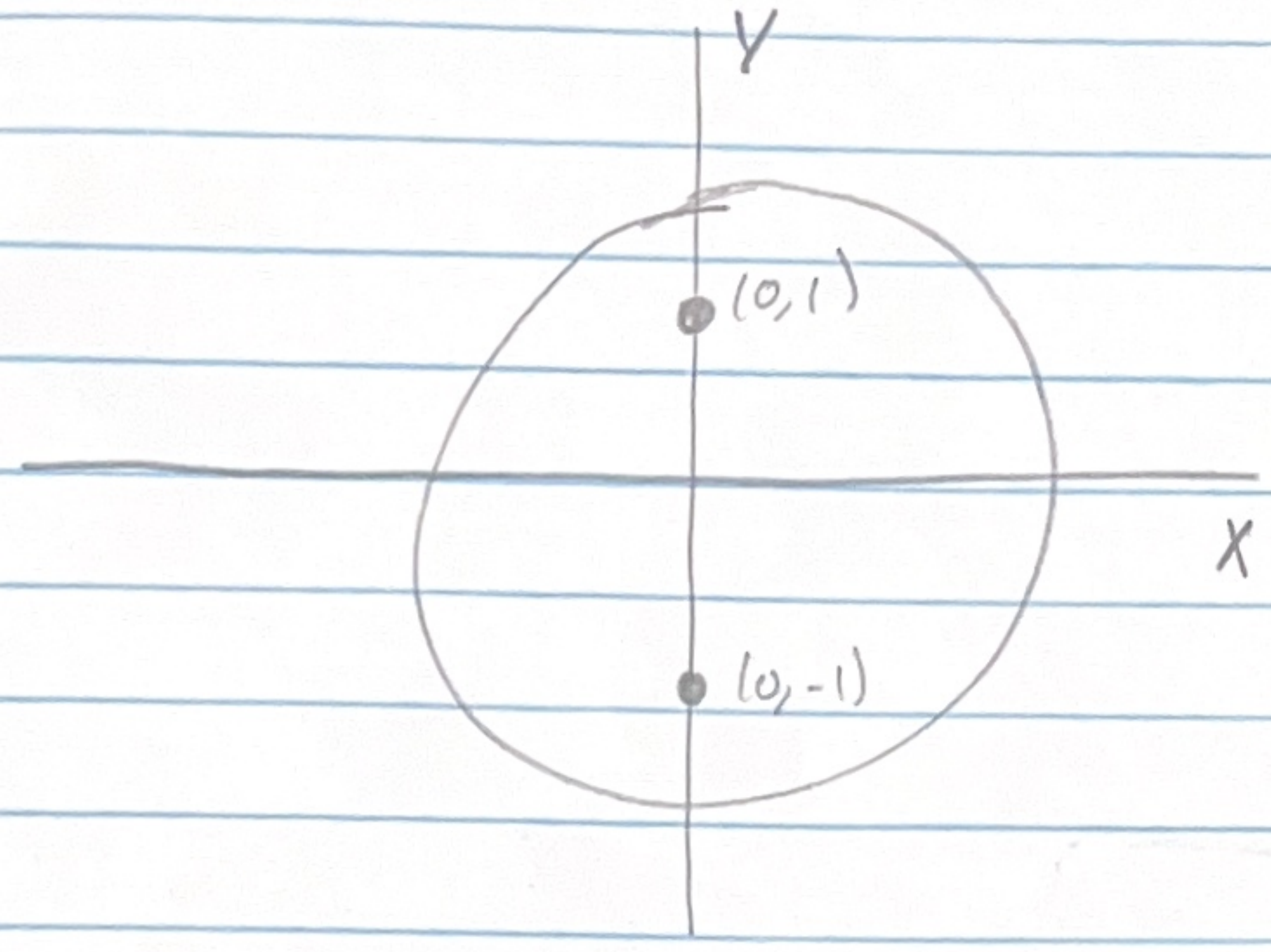
local max Saddle point

$f(1,1) = 1$
$f(0,0) = 0$
$f(0,2) = 0$
$f(2,0) = 0$
$f(2,2) = 0$



$f(x,y) = 0$

says $\begin{cases} x = 0 & y = 0 \\ x = 2 & y = 2 \end{cases}$ either



$f_{xx} = 6xy^2$
$f_{yy} = 2x^3 - 24y^2x$
$f_{xy} = 6x^2y - 8y^3$

$D = f_{xx}f_{yy} - f_{xy}^2$ @ $(0,1), (0,-1)$

$D(0,\pm 1) < 0$ ∴ Saddle

No max/min on Interior so Find max/min on the border subject to the constraint $x^2+y^2=4$

one variable problem

$f(x,y) = x^3y^2 - 2y^4x + 2x$

$\boxed{g(x) = x^3(4-x^2) - 2(4-x^2)^2x + 2x, \ -2 \le x \le 2}$

* used Substitution from constraint $y = 4-x^2$

$g(x) = x^3(4-x^2) - 2(4-x^2)^2x + 2x$

$g(x) = 4x^3 - x^5 - 32x + 16x^3 - 2x^5 + 2x$

$= -3x^5 + 20x^3 - 30x$

$g'(x) = -15x^4 + 60x^2 - 30 = 0$

$= -15(x^4 - 4x^2 - 2) = 0$

$x^2 = \dfrac{4 \pm \sqrt{16-8}}{2}$

$x^2 = 2 \pm \sqrt{2}$

$x = \pm\sqrt{2 \pm \sqrt{2}}$ 4 Sign combos = 4 solutions

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Optimization

Use Lagrange Multipliers:

Want to maximize $f(x,y)$ subject to constraint $g(x,y)=0$

How:

f is optimized where ∇f is $\parallel$ to ∇g

So we look for a place (x_0,y_0) where

$$(\nabla f)(x_0,y_0) = \lambda(\nabla g)(x_0,y_0)$$

we solve
$$\left.\begin{aligned} f_x &= \lambda g_x \\ f_y &= \lambda g_x \\ g &= 0 \end{aligned}\right\}$$
3 eqn's 3 unknowns x, y, λ

Same for 3 variables:

we solve
$$\left.\begin{aligned} f_x &= \lambda g_x \\ f_y &= \lambda g_y \\ f_z &= \lambda g_z \\ g &= 0 \end{aligned}\right\}$$
4 eqn's 4 unknowns x, y, z, λ

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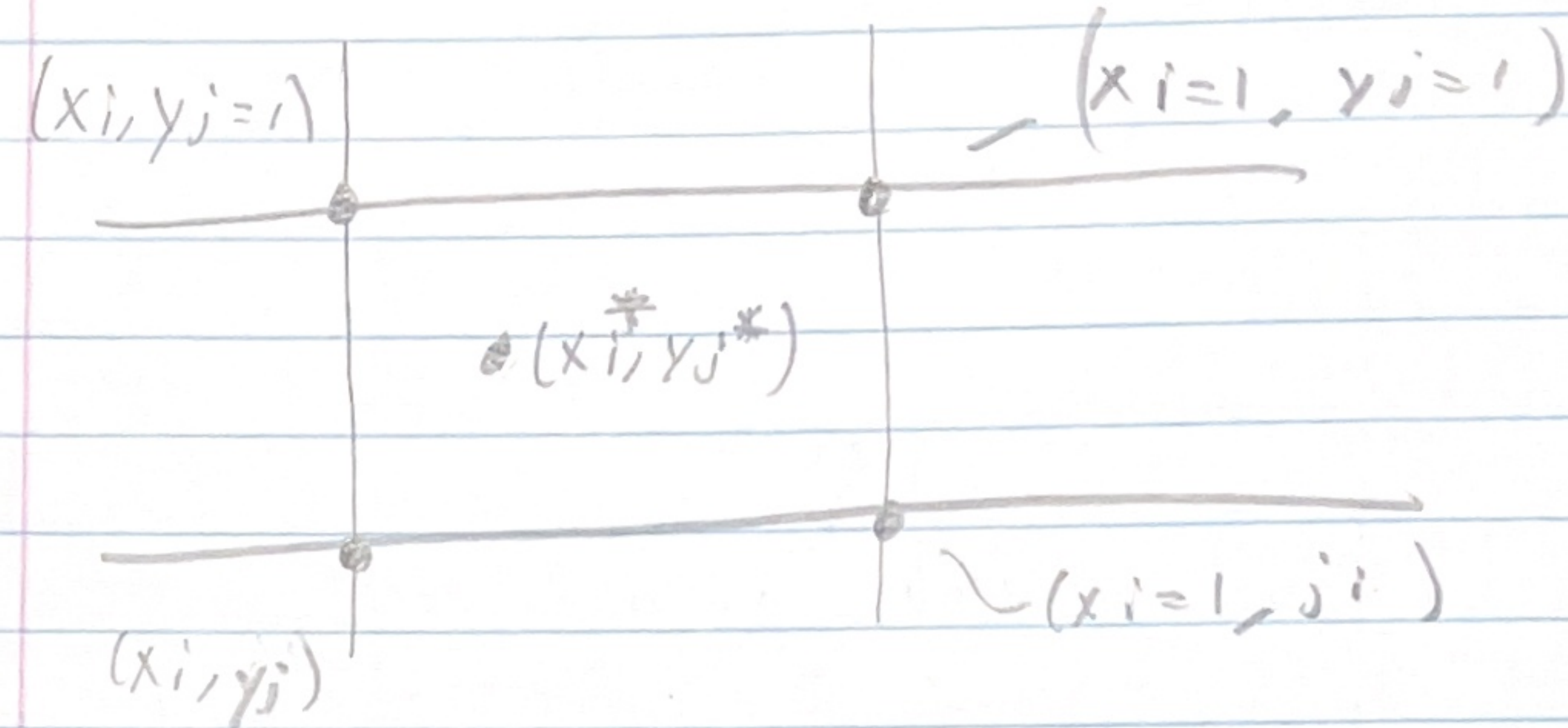
Double Integrals



Iterated integrals

$$\iint_R f(x,y)\,dA = \int_{y=c}^{d}\int_{x=a}^{b} f(x,y)\,dx\,dy$$

Fubini thm:

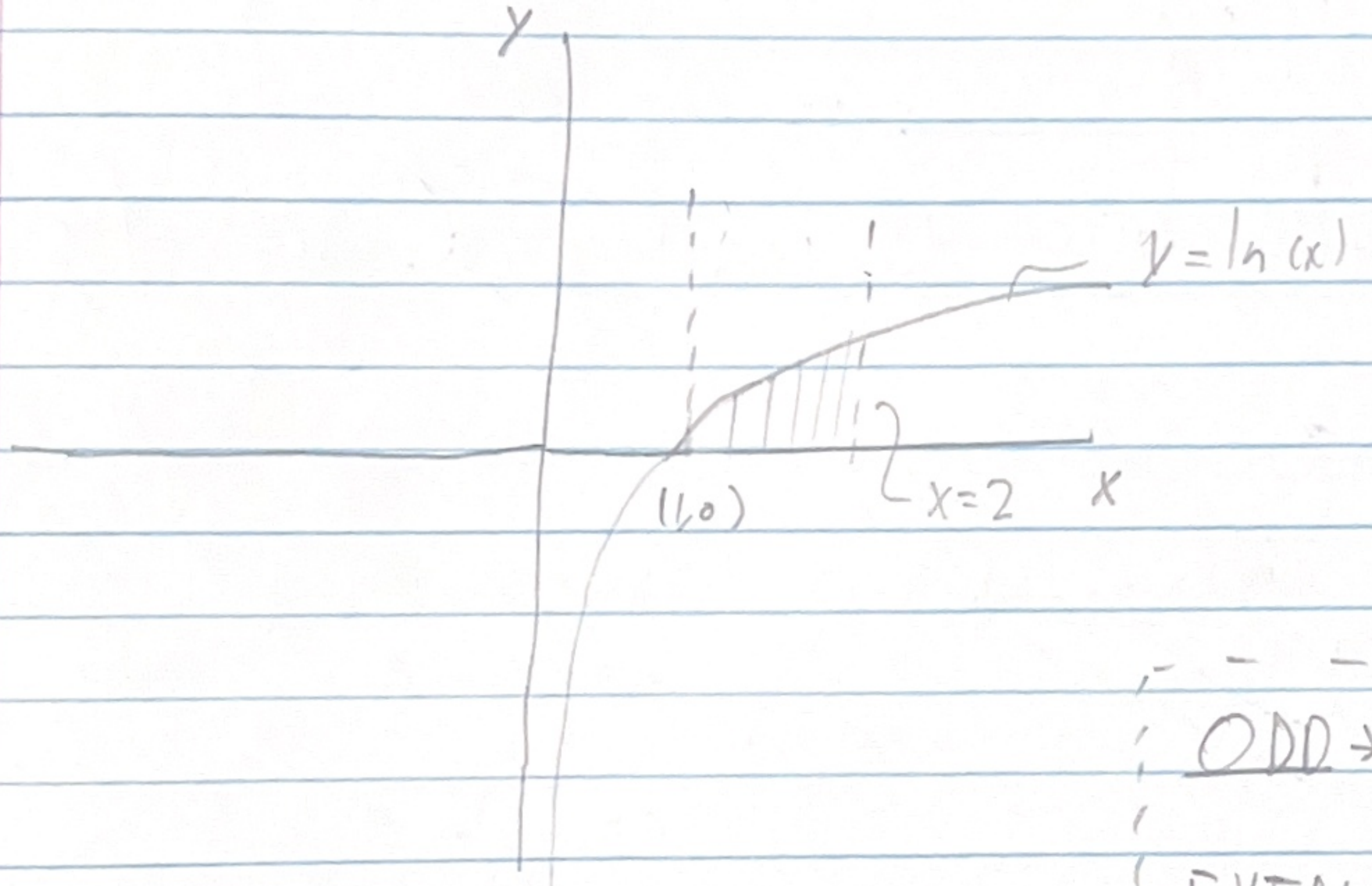
Says order doesn't matter like partial derivatives $f_{xy} = f_{yx}$

ie: $$\int_{y=c}^{d}\int_{x=a}^{b} f(x,y)\,dx\,dy = \int_{x=a}^{b}\int_{y=c}^{d} f(x,y)\,dy\,dx$$

Example

Reverse the order of integration for the following

$$\int_1^2 \int_0^{\log(x)} f(x,y)\,dy\,dx$$



ODD → $f(-t) = -f(t)$

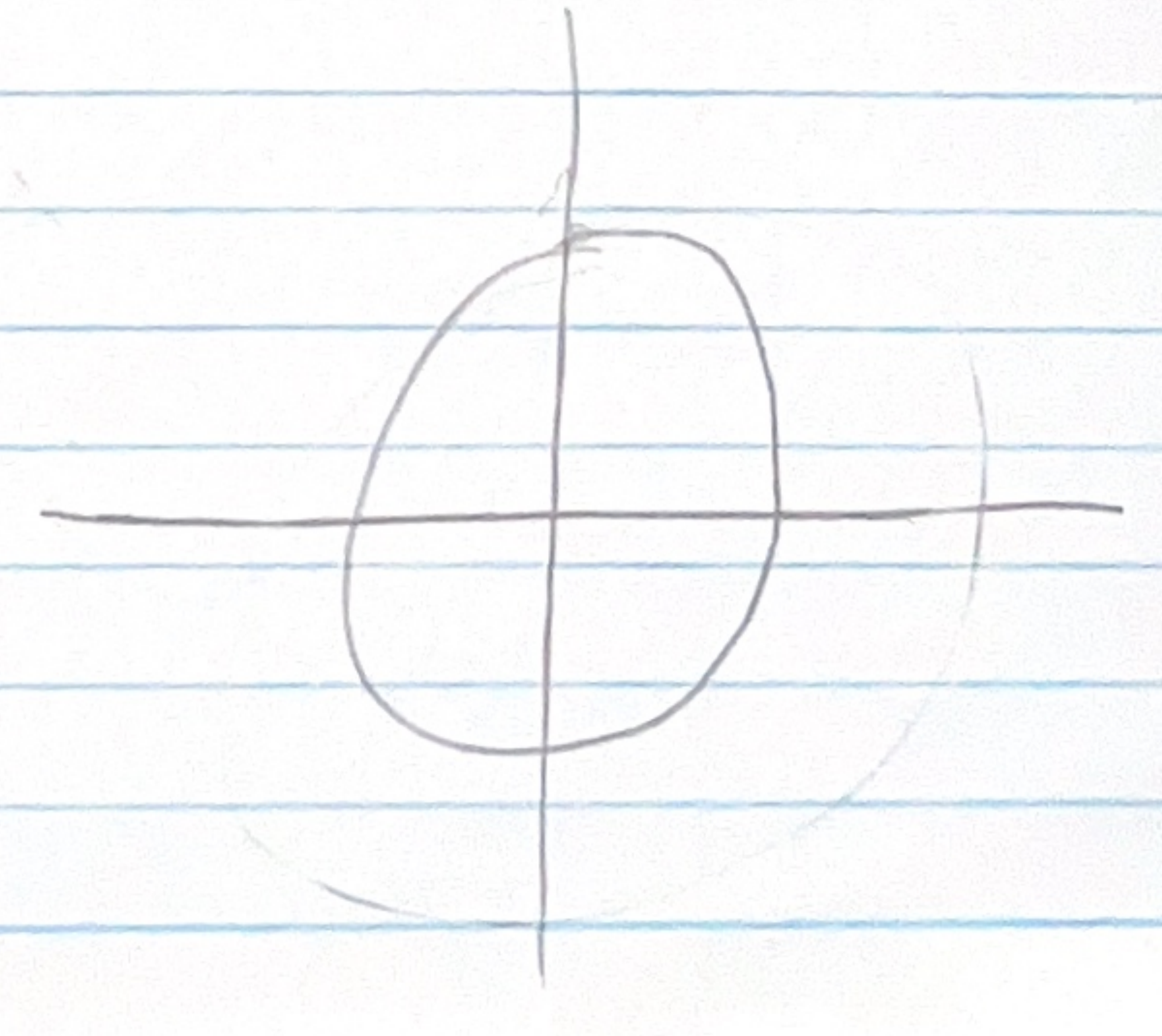
EVEN → $f(-x) = f(x)$

$$\int_0^{\log(2)} \int_{e^y}^{2} f(x,y)\,dx\,dy$$

Other tricks → Symmetry

let $R = \{x^2+y^2 \leq 2\}$

find $\iint_R x^2\tan(x) + \sin y^3 + 5\,dA$



$\iint_R f(r,\theta)\,dA$ as a Riemman Sum

$$\Delta r = \frac{b-a}{N}$$

$$\Delta\theta = \frac{\delta-\alpha}{M}$$

Area of  one of these blocks

$r_i\Delta\theta$ Δr

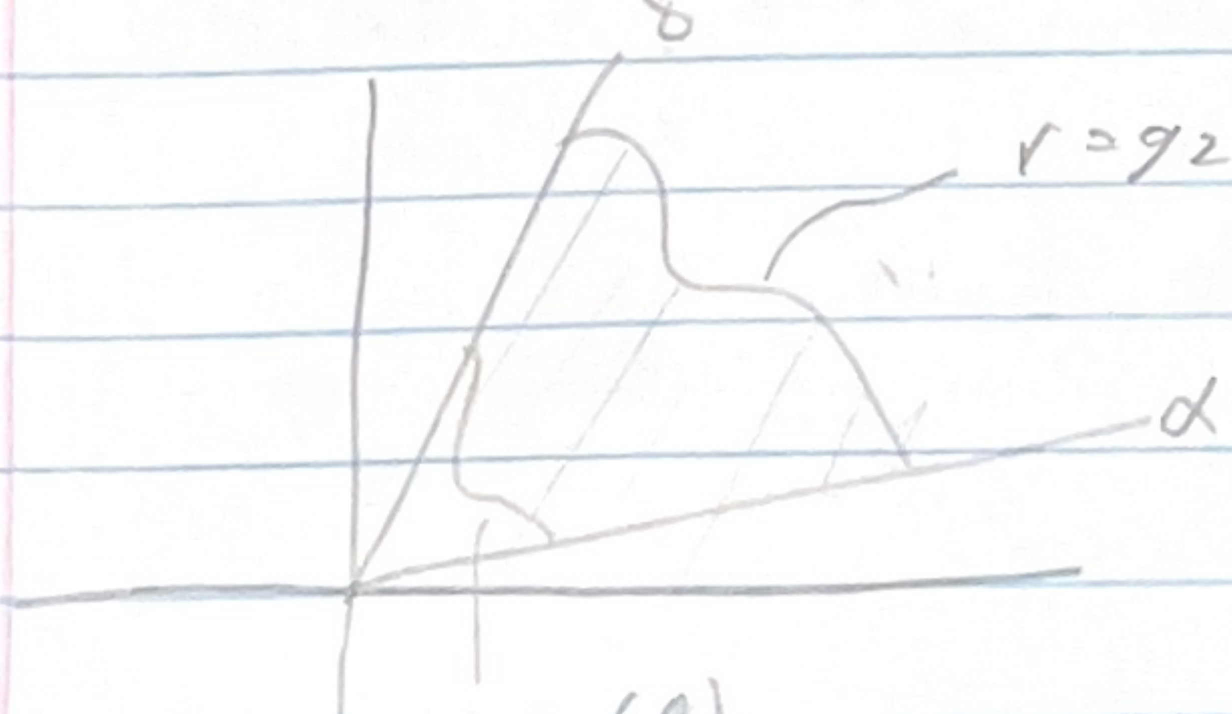
$\therefore$ Area $= r_i\Delta\theta\Delta r$

$\boxed{\text{Formula}}$

$$\iint_R f(r,\theta)\,dA = \int_{r=a}^{b}\int_{\theta=\alpha}^{\delta} f(r,\theta)\,r\,d\theta\,dr$$

$\boxed{\text{Region types in Polar Coordinates}}$

Type 1:



$$\int_{\theta=\alpha}^{\delta}\int_{r=g_1(\theta)}^{g_2(\theta)} f(r,\theta)\,r\,dr\,d\theta$$

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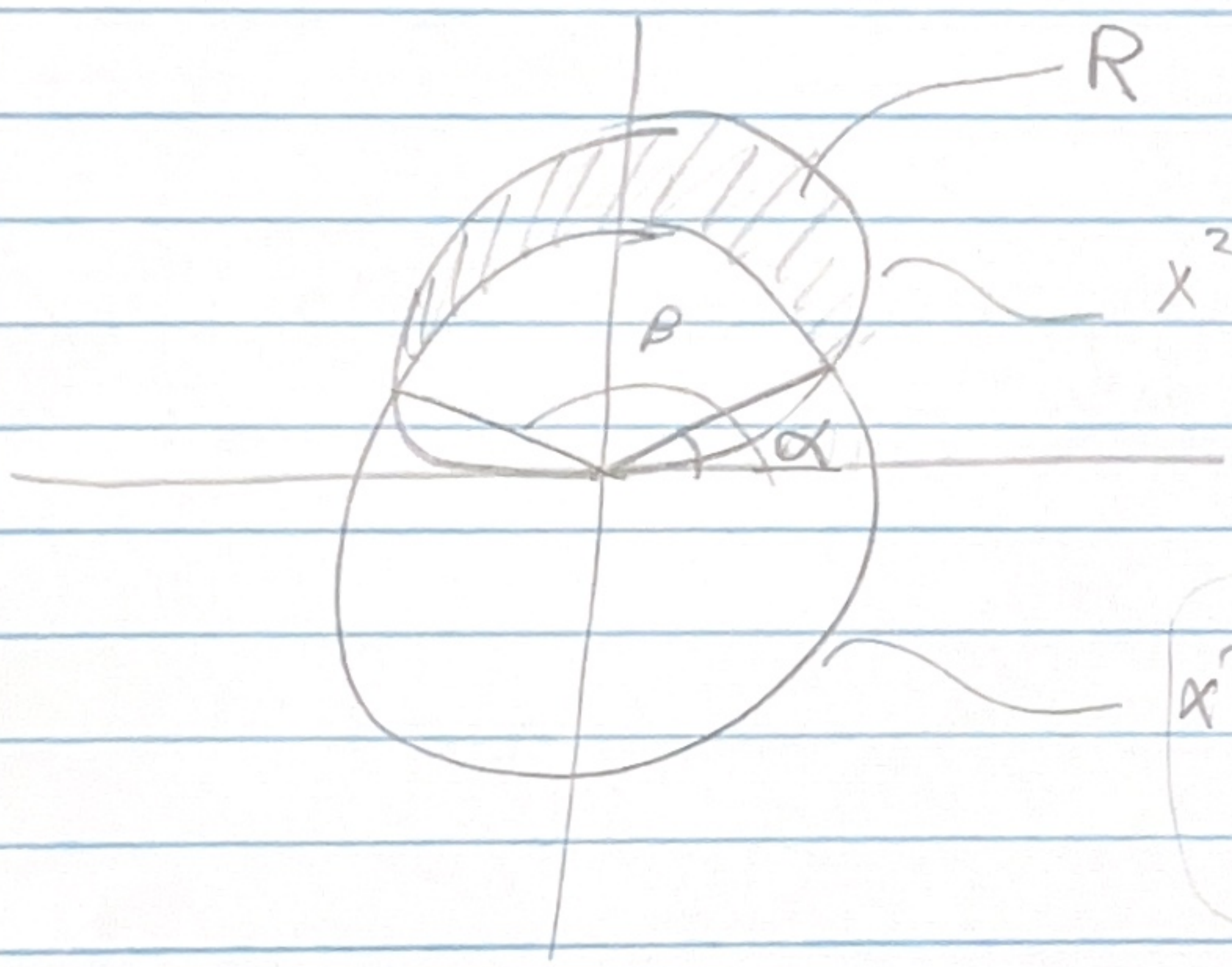
Polar Coordinates

Example Find the area of the region outside the circle of radius $\sqrt{2}$ centered at origin and inside the circle of radius 1 centered at (0,1)

sol'n:



$x^2+(y-1)^2=1$

$x^2+y^2=2 = r^2=2$
$r=\sqrt{2}$

$$\text{Area} = \iint_R 1\, dA$$

$$= \boxed{\int_{\theta=\frac{\pi}{4}}^{\frac{3\pi}{4}} \int_{r=\sqrt{2}}^{2\sin\theta} r\, dr\, d\theta}$$

$x^2+y^2-2y+1=1$
$x^2+y^2=2y$
$r^2=2r\sin\theta$
$r=2\sin\theta$

Intersect at $(1,1), (-1,1)$

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Triple Integrals

Double Integral Applications

$\iint_R f(x,y)\,dA \rightarrow$ Volume under graph $z=f(x,y)$; $f>0$

$\iint_R 1\,dA \rightarrow$ Area(R)

$\iint_R \rho(x,y)\,DA \rightarrow$ Total population if ρ is population density
Total mass if ρ is mass density

$\iint_R \sqrt{1+f_x^2+f_y^2}\,dA \rightarrow$ Surface area of $z=f(x,y)$

Triple integrals

$f(x,y,z)$

E is a solid region in 3-space

$$\iiint_E f(x,y,z)\,dV$$